Stackelberg and Nash Equilibrium Computation in Non-Convex Leader-Follower Network Aggregative Games

Rongjiang Li, Guanpu Chen, Die Gan, *Member, IEEE*, Haibo Gu, *Member, IEEE*, and Jinhu Lü, *Fellow, IEEE*

***Abstract*—This paper considers Stackelberg equilibrium (SE) and Nash equilibrium (NE) computation in a class of non-convex network aggregative games with one leader and multiple followers. The cost function of each follower is influenced by its strategy, the leader's strategy, and its neighbors' aggregative strategies. Also, the structured non-convex cost function of the leader is the composition of a canonical function and a vector-valued geometrical operator that relies on its strategy and followers' strategies. In the leader-follower scheme, when the leader has knowledge of the best responses of the followers in a closed form, the SE strategy will be the optimal choice due to its relatively low cost. When the leader does not know the exact expression of followers' best responses or the leader's dominance is threatened, NE will be what all players are committed to achieving. The widespread existence of nonconvexity creates a significant challenge for computing the above equilibria in different circumstances. The results in existing convex games are not directly applicable to such a non-convex case, as they get trapped in local equilibria or stationary points rather than global equilibria. Here, we adopt the canonical transformation to reformulate the non-convex games and present the existence condition based on the canonical duality theory. Then two projection gradient algorithms are designed to pursue the SE and the NE, followed by proving the convergence of the algorithms.**

***Index Terms*—Non-convex, leader-follower game, Stackelberg equilibrium, Nash equilibrium, network aggregative game.**

I. Introduction

MULTI-AGENT systems consisting of multiple interacting intelligent agents are widespread in nature and engineering, and have been extensively studied using control, optimization, and game theory techniques [1], [2], [3], [4]. In recent years, there has been tremendous interest in hierarchical decision-making structures in multi-agent systems with competitive agents. Here, the agents (players) are clustered into two groups, leaders and followers, depending on their roles in the game or on the types of their cost functions. Generally, the leaders commit to their strategies before the followers react to the strategies made by the leaders. Such leader-follower games emerge in various fields, such as mobile blockchain mining [5], unmanned aerial vehicle networks [6], and smart grids [7]. In particular, when the cost of each player is influenced by other players only through the aggregation behavior of their strategies, this framework is known as network aggregative game, which has a spectrum of practical applications, including cyber-physical systems [8], demand response management [9], and wireless cellular networks [10]. Aggregative games can simplify mathematical analysis and reduce computational complexity, especially for large-scale systems. Moreover, since aggregative games are characterized by an individual-aggregate interaction, they fit particularly well into the framework of one leader and multiple followers.

Under the most general circumstances involving one leader and multiple followers, the leader chooses a strategy to optimize its cost by taking into account the possible responses of followers, while followers observe this strategy of the leader and subsequently respond optimally to it. The corresponding equilibrium in this paradigm is the celebrated Stackelberg equilibrium (SE) [11]. Several studies have investigated the search for SE in leader-follower games. For instance, reference [12] formulated a Stackelberg game to investigate the secrecy rate maximization problem and analytically derived the SE. Reference [13] recast the Stackelberg game as a mathematical program with complementarity constraints and adopts the sequential convex approximation to arrive at the local SE point.

Besides, there are other common scenarios where the leader is unable to obtain the best responses of the followers, or the leader's dominance is threatened by uncertainties in the environment, errors in transmission in security games, etc [14], [15], [16], [17], [18]. In view of this point, NE is more appropriate to characterize the strategic interactions among the leader and followers. Here, NE refers to the joint strategy that no player can gain by unilaterally deviating from its current strategy. Some studies aim to find NE in leader-follower games. For instance, reference [14] investigated the search

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Rongjiang Li and Guanpu Chen are with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, and also with the School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China (e-mail: lirongjiang@amss.ac.cn; chengp@amss.ac.cn).

Die Gan is with the Zhongguancun Laboratory, Beijing 100094, China (e-mail: gandie@amss.ac.cn).

Haibo Gu and Jinhu Lü are with the School of Automation Science and Electrical Engineering, Beihang University, Beijing 100191, China, and also with the Zhongguancun Laboratory, Beijing 100094, China (e-mail: guhaibo@amss.ac.cn; jhlu@iss.ac.cn).

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for the NE when the leader faces uncertainty about the follower's surveillance capacity in security games. Reference [19] designed a distributed subgradient algorithm that achieves NE in leader-follower games after dealing with followers' stochastic activeness and communication.

Note that in the existing literature on multi-player games, the cost functions of players are usually assumed to be convex, strict convex, or strongly convex (see, e.g., [19], [20], [21], [22], [23]). In this case, the equilibrium problem is solvable, and the corresponding analytical and algorithmic tools are relatively well-established [24]. However, it is worth mentioning that nonconvexity is common in the equilibrium problem, which has recently aroused interest in such fields as market analysis [25], signal processing [26], and machine learning [27]. Nevertheless, it is challenging for the non-convex case to reach the SE and NE. On the one hand, tools that are effective for convex assumptions are not necessarily applicable to the non-convex case. For example, the classical gradient-based methods for convex games often get stuck in local equilibria when tracking the pseudo-gradients. On the other hand, non-convex problems have different structures and are too complicated to solve within one uniform framework. Actually, many reformulations and algorithms were designed and performed to circumvent the difficulties posed by nonconvexity. Reference [27] introduced a gradient-based Nikaido-Isoda function and proved error bounds to a Nash point. Reference [28] presented a gradient-proximal algorithm for approximate NE within non-convex aggregative games. Without the guarantee of convexity, [27] and [28] only reach the so-called local NE instead of the NE. To find the global minimum, stochastic annealing algorithms were introduced (see, e.g., [29], [30]), which added the additional greedy factors in each agent's update to escape from local optima in probability. As heuristic algorithms, however, they lack theoretical analysis and are time-consuming. Therefore, finding equilibria in a typical class of non-convex games is still a complex and challenging task that deserves further research.

The above facts motivate this paper to compute the SE and NE in a typical class of non-convex network aggregative games, including one leader and multiple followers. Compared with the related literature, our contributions are summarized as follows:

- Existing works [19], [20], [21], [22], [23] revolved around the equilibrium problem in convex games, while [13], [27], and [28] dealt with non-convex games and reached local equilibria or stationary points. To reach equilibrium in non-convex games, we explore a typical class of non-convex games where the leader's cost function is composited with a canonical function and a vector-valued geometrical operator. We utilize the canonical duality theory [31], [32], [33] to deal with the difficulties caused by nonconvexity and obtain the existence condition.
- In the case where the leader is accessible to followers' best response strategies in the closed form, the leader gives preference to the strategy in the SE point. We first reformulate the SE computation as an optimization problem in the primal-dual framework according to the canonical duality theory. We then propose a projected gradient algorithm under a constant step size setting. Different from [29] and [30], which only give numerical simulations and lack comprehensive theoretical analysis, we provide proof of the convergence of our algorithm to the SE.
- In cases where the leader has no access to the explicit solution of the best responses of the followers or the leader loses its dominant position, the players turn to the search for NE. Compared with works [31], [32] concentrating on optimization models, we consider the mutual coupling of the players' strategies and apply canonical duality theory to game models. Afterward, we propose a projected gradient algorithm where the leader's strategy is updated only at specific iteration steps due to the restricted communication between the leader and the followers. Also, the convergence of our algorithm to the NE is established.

An outline of this paper is as follows. Section II outlines some basic preliminaries. Section III gives the problem formulation of this paper to follow up the discussion on how to deal with the dilemma of choosing SE and NE. The computation of SE and NE is provided in Section IV and Section V, respectively. Simulation example is provided in Section VI. Finally, the concluding remarks are made in Section VII.

II. Preliminaries

In this section, we give some notations and preliminary knowledge.

A. Notations

$\mathbb{R}^n$ represents the set of n-dimensional real vectors. $\mathbb{R}^{m\times n}$ is the set of real matrices with m rows and n columns. I_m stands for the m-dimensional square identity matrix. For a matrix $A \in \mathbb{R}^{m\times n}$, A^T denotes the transpose of matrix A. The notation $\mathrm{col}\{\cdot, \cdots, \cdot\}$ is used to denote the stacked vector or matrix, and $\mathrm{diag}(\cdot, \cdots, \cdot)$ is used to denote a block matrix formed in a diagonal manner of the corresponding vectors or matrices. The Kronecker product of two matrices A and B is denoted by $A \otimes B$. For two real symmetric matrices $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$, $X \succ Y$ ($X \succeq Y$) means that $X - Y$ is a positive definite (semi-definite) matrix. $\Omega_1 \times \cdots \times \Omega_N$ denotes the Cartesian product of the sets $\Omega_1, \cdots, \Omega_N \subset \mathbb{R}^n$.

B. Convex Analysis

A set $\Omega \subset \mathbb{R}^n$ is convex, if $\lambda x_1 + (1-\lambda)x_2 \in \Omega$, $\forall x_1, x_2 \in \Omega$, $\forall\ 0 \le \lambda \le 1$ holds. Let $\Omega \in \mathbb{R}^n$ be a nonempty and convex set, then the mapping $P_\Omega : \mathbb{R}^n \to \Omega$ is defined by $P_\Omega(x) = \arg\min_{y\in\Omega} \|y - x\|^2$, where $\|\cdot\|$ represents Euclidean norm. A useful property closely related to the projection operation is its nonexpansiveness, i.e., $\|P_\Omega(x) - P_\Omega(y)\| \le \|x - y\|, \forall x, y \in \mathbb{R}^n$. Let Ω be a convex set and $x \in \Omega$, then the set defined as follows is called the normal cone of Ω at x

$$\mathcal{N}_\Omega(x) = \left\{ d \mid d^T(y - x) \le 0 \text{ for any } y \in \Omega \right\}.$$

A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is said to be σ-strongly convex for a given $\sigma > 0$, if for any $x, y \in \Omega$,

$$(\nabla f(x) - \nabla f(y))^T (x - y) \ge \sigma \parallel y - x \parallel^2$$

hold, where $\nabla f(\cdot)$ is the gradient of f and σ is called the strong convexity parameter. Moreover, let $L \ge 0$, the function

f is said to be L-smooth, if it is differentiable over Ω and satisfies

$$\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \| \text{ for all } x,\ y \in \Omega.$$

Saddle point is a point $(\bar{x}, \bar{y}) \in \Omega_x \times \Omega_y$ s.t. $f(\bar{x}, y) \leq f(\bar{x}, \bar{y}) \leq f(x, \bar{y})$, $\forall x \in \Omega_x$, $\forall y \in \Omega_y$.

C. Canonical Duality Theory

To address the challenges posed by the nonconvexity of cost functions, the canonical duality theory [31] is introduced.

A differentiable function $\Phi : \Xi \to \mathbb{R}$ is said to be a canonical function on Ξ, if its gradient $\nabla\Phi : \Xi \to \Xi^*$ is a one-to-one mapping from Ξ to its range Ξ^*. The canonical functions include a large class of import functions, for example, exponent functions $f(x) = e^x$, strict convex quadratic functions $f(x) = \frac{1}{2}x^T Ax + b^T x + c$ with $A \succ 0$, negative log functions $f(x) = -\log(x)$ [34], [35], [36]. Besides, if Φ is a convex canonical function, its Legendre conjugate $\Phi^* : \Xi^* \to \mathbb{R}$ can be defined uniquely by the Legendre transformation

$$\Phi^*(\eta) = \left\{\xi^T \eta - \Phi(\xi) | \eta = \nabla\Phi(\xi),\ \xi \in \Xi\right\},$$

where $\eta \in \Xi^*$ is a canonical dual variable. Then the canonical duality relations

$$\eta = \nabla\Phi(\xi) \Leftrightarrow \xi = \nabla\Phi^*(\eta) \Leftrightarrow \Phi(\xi) + \Phi^*(\eta) = \xi^T \eta \quad (1)$$

hold on $\Xi \times \Xi^*$, and (ξ, η) is called the Legredre canonical duality pair on $\Xi \times \Xi^*$.

Next, we review some lemmas needed in the following sections.

Lemma 1 ([34]): For a given constant $\sigma > 0$, we have

(1) if $f : \Omega \to \mathbb{R}$ is a $\frac{1}{\sigma}$-smooth convex function, then f^* is σ-strongly convex with respect to the dual norm $\| \cdot \|^*$;

(2) if $f : \Omega \to (-\infty, \infty]$ is a proper closed σ-strongly convex function, then $f^* : \Omega^* \to \mathbb{R}$ is $\frac{1}{\sigma}$-smooth.

Lemma 2 ([37]): Let $\Omega \neq \emptyset$ be a closed set. Suppose that the map $F : \Omega \subset \mathbb{R}^n \to \Omega$ is a contraction with constant $L \in (0, 1)$. Then we have the following statements:

(1) The map F has a unique fixed point $x^\diamond$ in Ω.

(2) For any starting point $x^0 \in \Omega$, $x^{t+1} = F(x^t)$ generates a sequence $\{x^t\}$ converging to $x^\diamond$.

(3) For any sequence $\{x^t\}$ given in (2),

$$\|x^t - x^\diamond\| \leq \frac{L^t}{1 - L}\|x^0 - F(x^0)\|, \forall t \geq 1.$$

III. Problem Formulation

A. System Model

In this paper, we consider a hierarchical non-cooperative game with one leader and N followers. Let $\mathcal{V} \triangleq \{1, \cdots, N\}$ be the set of followers where each follower $i \in \mathcal{V}$ has its strategy (or decision variable) $x_i \in \Omega_i \subset \mathbb{R}^n$ and Ω_i is a nonempty convex set. Let $\boldsymbol{x} \triangleq \mathrm{col}\{x_1, \cdots, x_N\} \in \Omega \triangleq \Pi_{i=1}^N \Omega_i \subset \mathbb{R}^{nN}$ be the strategy profile of all followers. The followers are connected with each other via a weighted directed graph $G(\mathcal{V}, \mathcal{A})$. Here, $\mathcal{A} = \left[a_{ij}\right]_{i,j\in\mathcal{V}}$ is the weighted adjacency matrix whose elements satisfy $a_{ij} > 0$ if there is a communication link from follower j to follower i and $a_{ij} = 0$ otherwise. The neighbors of follower i are denoted by $\mathcal{N}_i \triangleq \{j \in \mathcal{V} | a_{ij} > 0\}$. We assume that $\sum_{j\in\mathcal{V}\setminus\{i\}} a_{ij} = 1$.

Each follower i attempts to minimize a quadratic cost function $J_i^F(x_i, \sigma_i(\boldsymbol{x}_{-i}), y)$ which is affected by its own decision variable x_i, the aggregate behavior of its neighbors $\sigma_i(\boldsymbol{x}_{-i}) = \sum_{j\in\mathcal{N}_i} a_{ij} x_j$ with $\boldsymbol{x}_{-i} = \mathrm{col}\{x_1, \cdots, x_{i-1},\ x_{i+1}, \cdots, x_N\}$, and also the leader's decision variable y which is selected from a convex and compact set denoted by $\Omega_y \subset \mathbb{R}^m$. The follower side game is given by

$$\begin{aligned}&\min_{x_i\in\Omega_i} J_i^F(x_i, \sigma_i(\boldsymbol{x}_{-i}), y)\\ &= \min_{x_i\in\Omega_i} \left\{\frac{1}{2}{x_i}^T Q_i x_i + (\sigma_i(\boldsymbol{x}_{-i}) + P_{0i}y)^T x_i\right\}, \quad (2)\end{aligned}$$

where $Q_i \succ 0$ and $P_{0i} \in \mathbb{R}^{n\times m}$.

Furthermore, the cost function of the leader is defined as $J^L(y, \sigma_0(\boldsymbol{x}))$, where $\sigma_0(\boldsymbol{x}) = \sum_{i\in\mathcal{V}} a_{0i}x_i$ is the aggregation term of all followers. Here a_{0i} represents the weight of communication link between follower i and the leader, and satisfies $\sum_{i\in\mathcal{V}} a_{0i} = 1$ and $a_{0i} \geq 0$. Denote the leader weight vector as $a_0 = (a_{01}, \ldots, a_{0n})^T$.

This paper focuses on a typical class of non-convex aggregative game problems in which the leader's cost function is well-structured. The leader side game can be expressed as:

$$\min_{y\in\Omega_y} J^L(y, \sigma_0(\boldsymbol{x})) = \min_{y\in\Omega_y} \Phi(\Lambda(y, \sigma_0(\boldsymbol{x}))), \quad (3)$$

where $\Lambda : \mathbb{R}^m \times \mathbb{R}^n \to \Xi \subset \mathbb{R}^p$ is a vector-valued geometrical operator such that $\Lambda = (\Lambda_1, \ldots, \Lambda_p)^T$ and $\Lambda_j : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is a quadratic function. $\Phi : \Xi \to \mathbb{R}$ is a convex differential and $\frac{1}{C}$-smooth canonical function.

Remark 1: Geometrically speaking, this non-convex structure may possess a certain symmetry, which usually leads to more than one global minimizer [31]. Thus, some literature [31], [32] includes quadratic and linear terms in (3) to break the tie and facilitate obtaining sufficient conditions for the existence of the global solution. Without loss of generality, we do not consider these terms in (3) since they have little effect on the steps of dealing with nonconvexity based on the canonical duality theory.

The non-convex cost function (3) has extensive application in many fields, such as sensor network localization [38], a large deformation elasticity problem [39], and robust neural network training [40]. Here, a specific practical example is provided for intuition about the above non-convex cost function.

Example 1: The log-posynomial function appearing in resource allocation [35], [36] is listed in the following:

$$\log\left(y^T Cy + y^T D\sigma_0(\boldsymbol{x})\right)^{-1}, \quad (4)$$

where y represents oligarch' transmitting resource, $\sigma_0(\boldsymbol{x})$ denotes the average transmitting resources of other companies, and matrices C, D represent the correlation coefficients. This function fits well with the above non-convex structures when we set $\Phi = \log(\Lambda)^{-1}$ and $\Lambda = y^T Cy + y^T D\sigma_0(\boldsymbol{x})$.

Accordingly, the non-convex network aggregative game with one leader and multiple followers is defined as follows:

$$\begin{cases} \text{Players}: \text{ followers } \mathcal{V} \text{ and the leader} \\ \text{Strategies}: \begin{cases} \text{Follower } i: x_i \in \Omega_i \\ \text{Leader}: \quad y \in \Omega_y \end{cases} \\ \text{Cost}: \begin{cases} \text{Follower } i: J_i^F\left(x_i, \sigma_i\left(\boldsymbol{x}_{-i}\right), y\right) \ (2) \\ \text{Leader}: \quad J^L\left(y, \sigma_0(\boldsymbol{x})\right) \ (3) \end{cases} \end{cases} . \tag{5}$$

B. Stackelberg Equilibrium

In this subsection, we introduce the Stackelberg equilibrium [11], a fundamental concept for analyzing the leader-follower scheme. Followers can observe the leader's actions and adjust their actions accordingly while the leader considers how followers will react when making decisions.

Consider the SE of the leader-follower game mentioned above as long as the leader can obtain the explicit expression of the optimal set of response strategies. Denote follower i's best response strategy set to the leader's strategy y by

$$BR_i(\boldsymbol{x}_{-i}, y) = \arg\min_{x_i \in \Omega_i} J_i^F\left(x_i, \sigma_i(\boldsymbol{x}_{-i}), y\right).$$

In general, the above best response strategy set relies on its neighbors' strategies since the followers communicate and play the non-cooperative game with each other. In the settings of our paper, the followers' best response (BR) strategy set to the leader's strategy can be compactly represented as,

$$BR(y) = \{\boldsymbol{x} \in \Omega | x_i \in BR_i(\boldsymbol{x}_{-i}, y)\}.$$

Definition 1 (Stackelberg Equilibrium): If there exists a single-valued map $BR(y): \Omega_y \to \Omega$ such that for all i and for any given $y \in \Omega_y$, $\boldsymbol{x}_{-i} \in \prod_{m \neq i} \Omega_m$,

$$J_i^F\left(BR_i(\boldsymbol{x}_{-i}, y), \sigma_i(\boldsymbol{x}_{-i}), y\right) \leq J_i^F\left(x_i, \sigma_i(\boldsymbol{x}_{-i}), y\right), \forall x_i \in \Omega_i$$

and there exists $y^\diamond \in \Omega_y$ such that

$$J^L(y^\diamond, \sigma_0(BR(y^\diamond))) \leq J^L(y, \sigma_0(BR(y))), \forall y \in \Omega_y,$$

then $(\boldsymbol{x}^\diamond, y^\diamond) \in \Omega \times \Omega_y$, with $BR(y^\diamond) = \boldsymbol{x}^\diamond$ is a Stackelberg Equilibrium point of the leader-follower game between the followers and leader.

Remark 2: It suffices to search for SE in the case where the followers' best response strategy set $BR(y)$ to the leader's strategy is supposed to be a single-valued map where followers have a unique best response to the leader's strategy. On the one hand, some simplification is reasonable since SE computation is an inherently ill-posed, non-convex problem. The uniqueness of equilibrium among followers under the fixed strategy of the leader is a common assumption to simplify the analysis for SE, such as [41], [42], and [43]. On the other hand, results can be readily generalized to the case in which $BR(y)$ is a multi-valued map by introducing the concepts of strong Stackelberg Equilibrium or weak Stackelberg Equilibrium [11], [44].

C. Nash Equilibrium

In some practical situations, the above leader-follower scheme may fail when the leader's dominance is threatened or the leader has no access to the best responses of followers [14], [15], [16], [17], [18]. In such cases, there is no longer a hierarchical decision structure between $N+1$ players, and all players have symmetric roles. Nevertheless, in order to distinguish the types of cost functions of players, we still retain the terms "leader" and "follower" as labels. The optimization problems for these two types of players are listed as (3) and (2), respectively. Here, the simultaneous-move game model and the corresponding solution concept Nash equilibrium may better reflect the actual situation [43]:

Definition 2 (Nash equilibrium): A pair of strategies $(\boldsymbol{x}^\diamond, y^\diamond)$ forms a Nash equilibrium if for all $i \in \mathcal{V}$,

$$\begin{aligned} &J_i^F(x_i^\diamond, \sigma_i(\boldsymbol{x}_{-i}^\diamond), y^\diamond) \leq J_i^F(x_i, \sigma_i(\boldsymbol{x}_{-i}^\diamond), y^\diamond), \ \forall x_i \in \Omega_i, \\ &J^L(y^\diamond, \sigma_0(\boldsymbol{x}^\diamond)) \leq J^L(y, \sigma_0(\boldsymbol{x}^\diamond)), \ \forall y \in \Omega_y. \end{aligned} \tag{6}$$

The above equation (6) means that each strategy in a NE is the best response to the other players' strategies in that equilibrium. Consequently, neither the leader nor the followers have anything to gain by changing only one's own strategy.

It is well known that the strategy in SE is more advantageous than that in NE from the leader's perspective [11], [43]. Nevertheless, when SE is not unavailable, NE also provides an acceptable lower bound utility. In both cases, the bumpy geometric structure of the non-convex problems hinders the search for the SE and NE. The previous techniques used for convex games, such as gradient descent algorithms, can become stuck in local equilibria or stationary points in non-convex cases since the stationary conditions of the players' optimization problems are no longer sufficient. In this paper, we aim to develop algorithms to compute SE and NE instead of their stationary or local counterparts.

IV. Stackelberg Equilibrium Computation

In this section, we consider the case where the leader has access to the explicit expressions about followers' best responses. The analysis of SE is generally carried out by the backward induction method to reflect the sequential dependence of decision-making. Here, the procedure for SE computation is given as follows:

Step 1. For a fixed leader's strategy, followers play a network aggregative game and solve a NE problem, i.e., each follower $i \in \mathcal{N}$ finds

$$BR_i(\boldsymbol{x}_{-i}, y) = \arg\min_{x_i \in \Omega_i} J_i^F\left(x_i, \sigma_i(\boldsymbol{x}_{-i}), y\right),$$

for any fixed $\boldsymbol{x}_{-i}$ and y. Then we obtain the followers' BR strategy set $BR(y)$ to the leader's strategy.

Step 2. The leader finds the best strategy $y^\diamond$ by solving the following composite non-convex optimization problem concerning y based on the canonical duality theory [31]:

$$\begin{aligned} y^\diamond &\in \arg\min_{y \in \Omega_y} J^L(y, \sigma_0(BR(y))) \\ &= \arg\min_{y \in \Omega_y} \Phi(\Lambda(y, (a_0^T \otimes I_n)BR(y))). \end{aligned} \tag{7}$$

Step 3. After the global minima $y^\diamond$ for (7) is obtained, followers' best responses are yielded,

$$\boldsymbol{x}^\diamond = BR(y^\diamond).$$

Then $(\boldsymbol{x}^\diamond, y^\diamond)$ forms the SE.

Specifically, we focus on a case that the followers' constraint sets $\Omega_i = \mathbb{R}^n$, for $i \in \mathcal{V}$ and the augmented matrix $Q = \mathrm{diag}(Q_1, \ldots, Q_N) + \mathcal{A} \otimes I_n$ is invertible. By Step 1,

$BR_i(\boldsymbol{x}_{-i}, y) = \{x_i \in \mathbb{R}^n \mid Q_i x_i + \sigma_i(\boldsymbol{x}_{-i}) + P_{0i} y = 0\}$. And in turn, we obtain the followers' best response strategy set to the leader's strategy $BR(y) = -Q^{-1}Py$ with $P = \mathrm{col}\{P_{01}, \cdots, P_{0N}\}$.

Form Step 2, the crucial step of SE computation, we aim to solve non-convex problem (7) for the leader and obtain the global solution. For this purpose, we employ canonical duality theory which enables us to eliminate the duality gap between the non-convex primal problem and the corresponding canonical dual problem [31].

We first perform the canonical transformation and derive a total complementary function to transform the non-convex game to a complementary dual problem. For convenience, denote

$$\xi = \Lambda(y, -(a_0^T \otimes I_n)Q^{-1}Py), \tag{8}$$

with its range being Ξ_1. By (1), we know $\Phi(\xi) = \xi^T\eta - \Phi^*(\eta)$, then the total complementary function defined in the canonical duality theory is as follows:

$$\begin{aligned}\Pi_1(y, \eta) &\triangleq \xi^T\eta - \Phi^*(\eta)\\ &= \eta^T\Lambda(y, -(a_0^T \otimes I_n)Q^{-1}Py) - \Phi^*(\eta).\end{aligned} \tag{9}$$

Since $\Pi_1(\cdot, \eta)$ is a quadratic function with respect to variable y, its Hessian matrix is y-free.

To obtain the global minimum in (7) according to the canonical duality theory, we introduce the set with respect to the canonical duality variable η as follows:

$$\mathcal{S}_1^+ = \left\{\eta \in \Xi_1 \mid \nabla_y^2\Pi_1(y, \eta) \succeq kI_m\right\},$$

where the constant $k > 0$.

Remark 3: From the definition of $\mathcal{S}_1^+$, it is clear that $\mathcal{S}_1^+$ is a convex set. By (8), we know ξ should be in a compact set since y comes from a compact set Ω_y. Then, η should also be in a compact set because of the continuity property of $\nabla\Phi$. Hence we know $\mathcal{S}_1^+$ is a compact set. The nonemptiness of $\mathcal{S}_1^+$ ensures the strong convexity of $\Pi_1(\cdot, \eta)$ with respect to y, and we denote the strong convexity parameter as C_1. Moreover, since $\Phi(\cdot)$ is $\frac{1}{C}$-smooth convex function, by Lemma 1, we know its conjugate function $\Phi^*(\cdot)$ is C-strongly convex. Therefore, $-\Pi_1(y, \cdot)$ is C-strongly convex over η.

The following theorem presents the existence of SE for the non-convex leader-follower game (5).

Theorem 1: If there exists $\eta^\diamond \in \mathcal{S}_1^+$ such that $(y^\diamond, \eta^\diamond)$ is the stationary point of $\Pi_1(y, \eta, \sigma_0(\boldsymbol{x}^\diamond))$ in (9), then $y^\diamond$ is the global minimizer of (7) and $(\boldsymbol{x}^\diamond, y^\diamond)$ is the SE of the non-convex network aggregative game (5).

Proof: Since $(y^\diamond, \eta^\diamond)$ is a stationary point of the total complementary function (9), then the following first-order condition holds:

$$\begin{aligned}0 &\in \nabla_y\Pi_1(y^\diamond, \eta^\diamond) + \mathcal{N}_{\Omega_y}(y^\diamond),\\ 0 &\in \nabla_\eta\Pi_1(y^\diamond, \eta^\diamond) + \mathcal{N}_{\Xi_1^*}(\eta^\diamond).\end{aligned}$$

That is

$$0 \in \left[\nabla_y\Lambda(y^\diamond, -(a_0^T \otimes I_n)Q^{-1}Py^\diamond)\right]^T\eta^\diamond + \mathcal{N}_{\Omega_y}(y^\diamond), \tag{10}$$

$$0 \in \Lambda(y^\diamond, -(a_0^T \otimes I_n)Q^{-1}Py^\diamond) - \nabla\Phi^*(\eta^\diamond) + \mathcal{N}_{\Xi_1^*}(\eta^\diamond). \tag{11}$$

Since $\nabla\Phi : \Xi_1 \to \Xi_1^*$ is a one-to-one mapping from its domain Ξ_1 to its range Ξ_1^*, (11) can be transformed into

$$\Lambda(y^\diamond, -(a_0^T \otimes I_n)Q^{-1}Py^\diamond) = \nabla\Phi^*(\eta^\diamond). \tag{12}$$

Using the duality relations (1), we can see that (12) is equivalent to

$$\eta^\diamond = \nabla\Phi(\Lambda(y^\diamond, -(a_0^T \otimes I_n)Q^{-1}Py^\diamond)). \tag{13}$$

By substituting (13) into (10) and using the chain rule of J^L, we obtain

$$\begin{aligned}&\left[\nabla_y\Lambda(y^\diamond, -(a_0^T \otimes I_n)Q^{-1}Py^\diamond)\right]^T\\ &\qquad \times \nabla\Phi(\Lambda(y^\diamond, -(a_0^T \otimes I_n)Q^{-1}Py^\diamond))\\ &= \nabla_y J^L(y^\diamond, \sigma_0(\boldsymbol{x}^\diamond)) = 0.\end{aligned}$$

Thus, $y^\diamond$ is a stationary point of J^L.

Due to the convexity of the function $\Phi(\cdot)$, its Legendre conjugate $\Phi^*(\cdot)$ is also convex. Then from (9), we can see that the total complementarity function $\Pi_1(y, \cdot)$ defined by (9) is concave with respect to the canonical dual variable η in $\Omega_y \times \mathcal{S}_1^+$. Besides, by the definition of $\mathcal{S}_1^+$, it is clear that $\mathcal{S}_1^+$ is a convex set and in relation to the variable y, $\Pi_1(\cdot, \eta)$ is convex in $\Omega_y \times \mathcal{S}_1^+$. In this light, we can obtain that $(y^\diamond, \eta^\diamond)$ is the global optimal point of Π_1 in $\Omega_y \times \mathcal{S}_1^+$, i.e.,

$$\Pi_1(y^\diamond, \eta) \le \Pi_1(y^\diamond, \eta^\diamond) \le \Pi_1(y, \eta^\diamond),\ \forall(y, \eta) \in \Omega_y \times \mathcal{S}_1^+.$$

This confirms that $y^\diamond$ is the global mimimum of (7). The proof is completed. □

Theorem 1 derives a sufficient condition for the SE point. It reveals that once the stationary point $(y^\diamond, \eta^\diamond)$ of $\Pi_1(y, \eta)$ is obtained, we can check whether $\eta^\diamond \in \mathcal{S}_1^+$ to identify the SE. Inspired by this, we attempt to design the projected dynamics for the leader's decision variable y and its canonical duality variable η via the stationary conditions of the total complementary function, which relies on a underlying assumption on $\mathcal{S}_1^+$ as follows:

Assumption 1: The set $\mathcal{S}_1^+$ is nonempty.

Remark 4: Assumption 1 means that these exists a canonical duality variable which is in Ξ_1 and satisfies $\nabla_y^2\Pi_1(y, \eta) \succeq kI_m$ at the same time. Assumption 1 guarantees that canonical duality theory is suitable for resolving the non-convex problem, which was similarly considered in classic optimization works (see e.g., [32], [45], [46], [47]). Once the game problem is defined, the set $\mathcal{S}_1^+$ can be constructed in an offline fashion. This computation procedure is actually not so hard in most practical cases, and we just need to find the intersection of its domain Ξ and the set where the canonical duality variable satisfies the semi-positive definiteness of the matrix.

For convenience, we denote $z = \mathrm{col}\{y, \eta\}$, $\mathcal{Z} = \Omega_y \times \mathcal{S}_1^+$, and the following continuous mapping:

$$\begin{aligned}F(z) &= \begin{bmatrix}\nabla_y\Pi_1(y, \eta)\\ -\nabla_\eta\Pi_1(y, \eta)\end{bmatrix}\\ &= \begin{bmatrix}(\nabla_y\Lambda(y, -(a_0^T \otimes I_n)Q^{-1}Py))^T\eta\\ -\Lambda\left(y, -(a_0^T \otimes I_n)Q^{-1}Py\right) + \nabla\Phi^*(\eta)\end{bmatrix}\\ &\triangleq \begin{bmatrix}d_y(y, \eta)\\ d_\eta(y, \eta)\end{bmatrix}.\end{aligned} \tag{14}$$

Algorithm 1

Input: step size τ.
Initialize: $y^0 \in \Omega_y$, $\eta^0 \in \mathcal{S}_1^+$.
for each time $t = 0, 1, 2, \cdots$ **do**
Leader:
The decision variable:
$y^{t+1} = P_{\Omega_y}(y^t - \tau d_y(y^t, \eta^t))$;
The canonical dual variable:
$\eta^{t+1} = P_{\mathcal{S}_1^+}(\eta^t - \tau d_\eta(y^t, \eta^t))$;
Followers:
$\boldsymbol{x}^{t+1} = -Q^{-1} P y^{t+1}$.
end for
Output: $\{\boldsymbol{x}^t\}, \{y^t\}$, $t = 1, 2, \cdots$

Now, we design an algorithm to search for the SE (see Algorithm 1).

Owing to the continuity of the gradient of $F(z)$ on the compact set $\mathcal{Z}$, the operator $F(z)$ is Lipschitz continuous and the corresponding Lipschitz constant is denoted as L. Then the convergence result of Algorithm 1 is obtained.

Theorem 2: Suppose that Assumption 1 holds.

(1) If the constant step size τ satisfies $\tau < \frac{2C_{\max}}{L^2}$ with $C_{\max} = \max(C, C_1)$, then Algorithm 1 converges;

(2) Moreover, if the convergent point $(y^\diamond, \eta^\diamond)$ is the stationary point of (9), then $(\boldsymbol{x}^\diamond, y^\diamond)$ is the SE of the non-convex network aggregative game (5).

Remark 5: The condition in Theorem 2 (2) is common in algorithm design based on the canonical duality theory [32]. A sufficient condition for SE is that the stationary point $(y^\diamond, \eta^\diamond)$ of $\Pi_1(y, \eta)$ satisfies $\eta^\diamond \in \mathcal{S}_1^+$. Inspired by this, we find the point that satisfies the above sufficient condition by projecting the iteration points onto the closed convex set $\mathcal{S}_1^+$, which makes the convergence point of the algorithm not necessarily the stationary point of (9) but may just fall on the boundary of the set. Hence, we need to verify whether the convergence point of the algorithm is the stationary point of (9).

Proof: We first prove that the operator $F(z)$ defined in (14) is strongly monotone. For any $z_1, z_2 \in \mathcal{Z}$,

$$\begin{aligned}
&(F(z_1) - F(z_2))^T (z_1 - z_2) \\
&= \left[d_y(y_1, \eta_1) - d_y(y_2, \eta_2)\right]^T (y_1 - y_2) \\
&\quad + \left[d_\eta(y_1, \eta_1) - d_\eta(y_2, \eta_2)\right]^T (\eta_1 - \eta_2) \\
&= [\nabla\Phi^*(\eta_1) - \nabla\Phi^*(\eta_2)]^T (\eta_1 - \eta_2) \\
&\quad + \eta_1^T (\xi_2 - \xi_1 + \nabla_y \xi_1^T (y_1 - y_2)) \\
&\quad + \eta_2^T (\xi_1 - \xi_2 - \nabla_y \xi_2^T (y_1 - y_2)) \\
&\geq C\|\eta_1 - \eta_2\|^2 + C_1\|y_1 - y_2\|^2 \\
&\geq C_{\max}\|z_1 - z_2\|^2,
\end{aligned}$$

where $\xi_i = \Lambda(y_i, -(a_0^T \otimes I_n) Q^{-1} P y_i)$, $i = 1, 2$. The inequality holds since $\Phi^*(\cdot)$ and $\eta^T \Lambda(y, -(a_0^T \otimes I_n) Q^{-1} P y)$ are strong convex with respect to η and y, respectively.

Furthermore, by the Lipschitz continuity of $F(z)$, we have for any $z_1, z_2 \in \mathcal{Z}$,

$$\begin{aligned}
&\|P_{\mathcal{Z}}(z_1 - \tau F(z_1)) - P_{\mathcal{Z}}(z_2 - \tau F(z_2))\|^2 \\
&\leq \|z_1 - z_2 - \tau(F(z_1) - F(z_2))\| \\
&= \|z_1 - z_2\| + \tau^2\|F(z_1) - F(z_2)\|^2 \\
&\quad - 2\tau[F(z_1) - F(z_2)]^T (z_1 - z_2) \\
&\leq (1 + \tau^2 L^2 - 2\tau C_{\max})\|z_1 - z_2\|^2.
\end{aligned}$$

Therefore by the condition $\tau < \frac{2C_{\max}}{L^2}$, we know $P_{\mathcal{Z}}(z - \tau F(z))$ is a contraction map. By Banach fixed-point theorem (see Lemma 2), for any starting point $z^0 = \text{col}\{y^0, \eta^0\} \in \mathcal{Z}$, the sequences $\{y^t\}$ and $\{\eta^t\}$ generated by Algorithm 1 converges to $y^\diamond$ and $\eta^\diamond$. Furthermore, from Theorem 1, the convergence of Algorithm 1 can be obtained. This accomplishes the proof. □

V. Nash Equilibrium Computation

In Section IV, we have analyzed the Stackelberg Equilibrium on the condition that the best responses of the followers to the leader's strategy can be expressed in a closed form. Nevertheless, the leader-follower scheme may sometimes fail since the leader may not obtain followers' best responses or may not maintain its dominant position [14], [15], [16], [17], [18]. For instance, in security games, intruders who act as followers occasionally take action directly because of the high surveillance costs of the defense strategy [17]. Similarly, attackers (followers) prefer to attack covertly rather than following the leader-follow scheme to get rid of the defender's (leader's) fault detection [18]. In such cases, the simultaneous-move game model may be considered an alternative where the leader and followers have symmetric roles. In other words, "leader" and "follower" are just two labels to distinguish the types of cost functions of players.

To consider the NE in this section, we assume that the strategy sets of followers $\Omega_i, i \in \mathcal{V}$ are compact.

A. The Derivation of NE

In this subsection, based on the canonical duality theory, we reformulate the NE computation as a complementary dual problem to explore the existence of NE with the following steps:

Step 1: We first make a canonical transformation and introduce the total complementary function $\Pi_2(y, \eta, \sigma_0(\boldsymbol{x}^\diamond))$. Then we consider the relationship between stationary points of $\Pi_2(y, \eta, \sigma_0(\boldsymbol{x}^\diamond))$, $J_i^F(x_i, \sigma_i(\boldsymbol{x}_{-i}), y)$ and NE point in Definition 2.

Step 2: A sufficient feasible domain $\mathcal{S}_2^+$ regarding the canonical dual variable is introduced to distinguish the NE from other stationary points.

Step 3: Existence condition of the NE is provided (see Theorem 3). The stationary point $(\boldsymbol{x}^\diamond, y^\diamond)$ of $\Pi_2(y, \eta, \sigma_0(\boldsymbol{x}^\diamond))$, $J_i^F(x_i, \sigma_i(\boldsymbol{x}_{-i}), y)$ such that $\eta^\diamond \in \mathcal{S}_2^+$ is the NE.

Following Section IV, we define the following total complementary function $\Pi_2(y, \eta, \sigma_0(\boldsymbol{x}))$ in a similar way as shown in (9) as follows:

$$\begin{aligned}
\Pi_2(y, \eta, \sigma_0(\boldsymbol{x})) &= \xi^T \eta - \Phi^*(\eta) \\
&= \eta^T \Lambda(y, \sigma_0(\boldsymbol{x})) - \Phi^*(\eta),
\end{aligned} \tag{15}$$

where $\xi = \Lambda(y, \sigma_0(\boldsymbol{x}))$ with its range being Ξ_2.

Note that $\Pi_2(\cdot, \eta, \sigma_0(\boldsymbol{x}))$ is a quadratic function with respect to variable y, its Hessian matrix is y-free and can be defined as

$$G(\eta, \sigma_0(\boldsymbol{x})) = \nabla_y^2 \Pi_2(y, \eta, \sigma_0(\boldsymbol{x})) = \sum_{k=1}^{q} [\eta]_k \nabla_y^2 \Lambda_k (y, \sigma_0(\boldsymbol{x}))$$

with $[\eta]_k$ being the k-th entry of the vector η. We also introduce the following set to prepare for NE computation:

$$\mathcal{S}_2^+ = \left\{\eta \in \Xi_2^* \mid G(\eta, \sigma_0(\boldsymbol{x})) \succeq k_{\boldsymbol{x}} I_n\right\}, \tag{16}$$

where the constant $k_{\boldsymbol{x}} > 0$.

Remark 6: In S_2^+, the positive definiteness of the Hessian matrix $G(\eta, \sigma_0(\boldsymbol{x}))$ is established for any fixed $\boldsymbol{x}$. Compared to S_2^+, S_1^+ is irrelevant to $\boldsymbol{x}$ (the strategy profile of all followers). This difference in sets reveals a significant difference between the SE and NE computation. Specifically, NE computation remains a variational problem, while SE computation is transformed into a non-convex optimization problem due to the definition of the best response set. Although most of the literature has considered similar non-convex structures in the framework of optimization [31], [32], it is not straightforward to transplant techniques in canonical duality theory from optimization problems to variational problems due to the mutual coupling of the players' strategies.

Then we get the following theorem which shows the relationship between stationary points of $\Pi_2(y, \eta, \sigma_0(\boldsymbol{x}^\diamond))$, $J_i^F(x_i, \sigma_i(\boldsymbol{x}_{-i}), y)$ and NE point in (6).

Theorem 3: If there exists $\eta^\diamond \in \mathcal{S}_2^+$ such that $(y^\diamond, \eta^\diamond)$ is the stationary point of $\Pi_2(y, \eta, \sigma_0(\boldsymbol{x}^\diamond))$ in (15) and $x_i^\diamond$ satisfies the basic first-order optimality condition of $J_i^F\left(x_i, \sigma_i(\boldsymbol{x}_{-i}^\diamond), y^\diamond\right)$, then $(\boldsymbol{x}^\diamond, y^\diamond)$ is the NE of the non-convex network aggregative game (5).

Proof: For a given strategy profile $(\boldsymbol{x}^\diamond, y^\diamond)$, if there exists $\eta^\diamond \in \Xi_2^*$ such that $(y^\diamond, \eta^\diamond)$ is the stationary point of $\Pi_2(y, \eta, \sigma_0(\boldsymbol{x}^\diamond))$, then it satisfies the following first-order conditions:

$$0 \in \left[\nabla_y \Lambda(y^\diamond, \sigma_0(\boldsymbol{x}^\diamond))\right]^T \eta^\diamond + \mathcal{N}_{\Omega_y}(y^\diamond).$$
$$0 \in -\Lambda(y^\diamond, \sigma_0(\boldsymbol{x}^\diamond)) + \nabla\Phi^*(\eta^\diamond) + \mathcal{N}_{\Xi_2^*}(\eta^\diamond).$$

Similar to the proof of Theorem 1, we have

$$0 \in \nabla_y J^L(y^\diamond, \sigma_0(\boldsymbol{x}^\diamond)) + \mathcal{N}_{\Omega_y}(y^\diamond). \tag{17}$$

Furthermore, by virtue of the first-order optimality condition, when $x_i^\diamond$ is the minimizer of the optimization problem

$$\min_{x_i \in \Omega_i} J_i^F\left(x_i, \sigma_i\left(\boldsymbol{x}_{-i}^\diamond\right), y^\diamond\right),$$

we have

$$0 \in \nabla_{x_i} J_i^F\left(x_i^\diamond, \sigma_i\left(\boldsymbol{x}_{-i}^\diamond\right), y^\diamond\right) + \mathcal{N}_{\Omega_i}(x_i^\diamond). \tag{18}$$

Since (18) holds for any $i \in \mathcal{V}$, $(\boldsymbol{x}^\diamond, y^\diamond)$ satisfies the Nash stationary condition.

Moreover, when $\eta \in \mathcal{S}_2^+$, it is clear that $\Pi_2(\cdot, \eta, \sigma_0(\boldsymbol{x}))$ is convex with respect to y. Since the conjugate function $\Phi^*(\cdot)$ is convex, then $\Pi_2(y, \cdot, \sigma_0(\boldsymbol{x}))$ is concave on η. In this light, we can obtain the globally optimality of $(y^\diamond, \eta^\diamond)$ on $\Omega_y \times \mathcal{S}_2^+$.

$$\Pi_2(y^\diamond, \eta, \sigma_0(\boldsymbol{x}^\diamond)) \le \Pi_2(y^\diamond, \eta^\diamond, \sigma_0(\boldsymbol{x}^\diamond)) \le \Pi_2(y, \eta^\diamond, \sigma_0(\boldsymbol{x}^\diamond)).$$

The inequality relation above tells us that given $\sigma_0(\boldsymbol{x}^\diamond)$,

$$J^L(y^\diamond, \sigma_0(\boldsymbol{x}^\diamond)) \le J^L(y, \sigma_0(\boldsymbol{x}^\diamond)), \forall y \in \Omega_y.$$

This confirms that $(\boldsymbol{x}^\diamond, y^\diamond)$ is the NE of (5), which completes the proof. □

Theorem 3 provides a possible method to compute the NE of the non-convex network aggregative game (5). However, it is usually tricky to obtain the explicit solution because of the complicated mutual coupling of stationary conditions. Thus, we will design a projected gradient algorithm in the following to allow the leader and followers to learn their optimal strategy simultaneously.

B. Algorithm Design

For the description of the algorithm, let's define some notations:

$$\begin{aligned} g_y(y, \eta, \sigma_0(\boldsymbol{x})) &\triangleq \nabla_y \Pi_2(y, \eta, \sigma_0(\boldsymbol{x})) \\ &= \left[\nabla_y \Lambda(y, \sigma_0(\boldsymbol{x}))\right]^T \eta, \\ g_\eta(y, \eta, \sigma_0(\boldsymbol{x})) &\triangleq -\nabla_\eta \Pi_2(y, \eta, \sigma_0(\boldsymbol{x})) \\ &= -\Lambda(y, \sigma_0(\boldsymbol{x})) + \nabla\Phi^*(\eta), \\ g_i(x_i, \sigma_i(\boldsymbol{x}_{-i}), y) &\triangleq \nabla_{x_i} J_i^F(x_i, \sigma_i(\boldsymbol{x}_{-i}), y). \end{aligned}$$

In view of the continuity of gradient of $g_i(x_i, \cdot, \cdot)$ on compact sets Ω_y and $\Omega_{\sigma_i} \triangleq \left\{\sigma_i(\boldsymbol{x}_{-i}) \mid \boldsymbol{x}_{-i} \in \prod_{m \neq i} \Omega_m\right\}$, $\forall i \in \mathcal{V}$, there exists Lipschitz constant L such that for all $i \in \mathcal{V}$, $\sigma_{i1}, \sigma_{i2} \in \Omega_{\sigma_i}$, we have

$$\begin{aligned} \| g_i(x_i, \sigma_{i1}, y_1) - g_i(x_i, \sigma_{i2}, y_2) \| &\le L \| \sigma_{i1} - \sigma_{i2} \| \\ &+ L \| y_1 - y_2 \|, \forall x_i \in \Omega_i, y_1, y_2 \in \Omega_y. \end{aligned} \tag{19}$$

Similarly, for any $y_1, y_2 \in \Omega_y$, $\eta_1, \eta_2 \in \Xi_2^*$, $\sigma_{01}, \sigma_{02} \in \Omega_{\sigma_0} \triangleq \{\sigma_0(\boldsymbol{x}) | \boldsymbol{x} \in \Omega\}$, there exists Lipschitz constant L_0 and L_1 such that

$$\begin{aligned} \|g_y(y, \eta_1, \sigma_{01}) - g_y(y, \eta_2, \sigma_{02})\| \le \quad & L_0\|\eta_1 - \eta_2\| \\ & + L_0\|\sigma_{01} - \sigma_{02}\|, \end{aligned} \tag{20}$$

$$\begin{aligned} \|g_\eta(y_1, \eta, \sigma_{01}) - g_\eta(y_2, \eta, \sigma_{02})\| \le \quad & L_1\|y_1 - y_2\| \\ & + L_1\|\sigma_{01} - \sigma_{02}\|. \end{aligned} \tag{21}$$

Here, assume that the leader does not update its strategy at each iteration but in a given set $\mathcal{T}^L \triangleq \{t_j^L\}_{j=0}^\infty$. Specifically, at iteration $t \in \mathcal{T}^L$, the leader computes its new strategy by taking a gradient step and then a projection to Ω_y, based on its previous strategy, the previous aggregate term of all followers' strategies, and the previous canonical dual variable.

$$y^{t+1} = \begin{cases} P_{\Omega_y}\left(y^t - \beta^t g_y(y^t, \eta^t, \sigma_0(\boldsymbol{x}^t))\right), & t \in \mathcal{T}^L \\ y^t, & t \notin \mathcal{T}^L \end{cases} \tag{22}$$

Similarly, canonical dual variable also utilizes the projected gradient method in the same iteration as the leader's decision variable.

$$\eta^{t+1} = \begin{cases} P_{\mathcal{S}_2^+}\left(\eta^t - \gamma^t g_\eta(y^t, \eta^t, \sigma_0(\boldsymbol{x}^t))\right), & t \in \mathcal{T}^L \\ \eta^t, & t \notin \mathcal{T}^L \end{cases} \tag{23}$$

To lower the communication cost, followers run multiple projected gradient steps to play network aggregative games until they receive a new update of the leader's decision variable.

$$x_i^{t+1} = P_{\Omega_i}\left(x_i^t - \alpha_i^t g_i(x_i^t, \sigma_i(\boldsymbol{x}_{-i}^t), y^t)\right), \quad \forall i. \tag{24}$$

Therefore, we get the following Algorithm 2.

Algorithm 2

Input: step sizes β^t, γ^t, α_i^t, $\forall i \in \mathcal{V}$.
Intialize: $y^0 \in \Omega_y$, $\eta^0 \in \mathcal{S}_2^+$, $x_i^0 \in \Omega_i$ for $\forall i \in \mathcal{V}$.
for each time $t = 0, 1, 2, \cdots$ **do**
  Leader:
    If $t \in \mathcal{T}^L$
      The decision variable:
        $y^{t+1} = P_{\Omega_y}(y^t - \beta^t g_y(y^t, \eta^t, \sigma_0(\boldsymbol{x}^t)))$;
      The canonical dual variable:
        $\eta^{t+1} = P_{\mathcal{S}_2^+}\left(\eta^t - \gamma^t g_\eta(y^t, \eta^t, \sigma_0(\boldsymbol{x}^t))\right)$;
    end if
  Follower $i \in \mathcal{V}$**:**
    $x_i^{t+1} = P_{\Omega_i}\left(x_i^t - \alpha_i^t g_i(x_i^t, \sigma_i(\boldsymbol{x}_{-i}^t), y^t)\right)$.
end for
Output: $\{x_i^t\}_{i=1}^N$, $\{y^t\}$, $t = 1, 2, \cdots$

Remark 7: We design the Algorithm 2 under a centralized framework. We acknowledge that the distributed algorithm has advantages over our centralized framework by eliminating requirements such as a central node for information broadcasting and full observation of players' strategies. Inspired by the literature review on distributed aggregative games in [48] and the distributed structure where the gossip-based communication protocol can be encountered as a special case in [19], it will be interesting to explore our framework under the distributed framework. However, introducing distributed frameworks has created challenges and difficulties in terms of communication and information exchange, consensus and convergence. Specifically, how to solve the problem of adding communication and convergence among agents without destroying the structure of the complementary function will be the focus of our following work.

Next, we will establish the convergence of Algorithm 2 to the NE within the non-convex network aggregative game (5).

C. Convergence Analysis

For further analysis, we introduce some assumptions.

Assumption 2: The set $\mathcal{S}_2^+$ defined in (16) is nonempty.

Remark 8: Similar to Remark 3, the nonemptiness of $\mathcal{S}_2^+$ ensures the strong convexity of $\Pi_2(\cdot, \eta, \sigma_0(\boldsymbol{x}))$ with respect to y, and we denote the strong convexity parameter as C_2. By Remark 3, we also know that $-\Pi_2(y, \cdot, \sigma_0(\boldsymbol{x}))$ is C-strongly convex over η. Assumption 2 are proposed to ensure the existence of the NE point.

Assumption 3: There exists a constant $\bar{T} < \infty$ such that $t_{j+1}^L - t_j^L \leq \bar{T}$ for any $j \in N$.

Assumption 4: Let β^t, γ^t, and α_i^t, $\forall i \in \mathcal{V}$ are nonincreasing, then $\sum_{t=0}^{\infty} \alpha_i^t = \infty$, and $\sum_{t=0}^{\infty} (\alpha_i^t)^2 < \infty$; $\sum_{t=0}^{\infty} \beta^t = \infty$, and $\sum_{t=0}^{\infty} (\beta^t)^2 < \infty$; $\sum_{t=0}^{\infty} \gamma^t = \infty$, and $\sum_{t=0}^{\infty} (\gamma^t)^2 < \infty$.

Assumption 5: There exists κ such that $\mu_{\max}^t \leq \kappa \mu_{\min}^t$, where $\mu_{\max}^t = \max(\alpha_1^t, \ldots, \alpha_N^t, \beta^t, \gamma^t)$ and $\mu_{\min}^t = \min(\alpha_1^t, \ldots, \alpha_N^t, \beta^t, \gamma^t)$.

Remark 9: Assumption 3 is about the iteration, which shows that with no more than $\bar{T}$ iterations, the leader's strategy and its dual variable will be updated. Assumption 4 reveals that step sizes are diminishing, which is commonly used in the existing literature (see e.g., [49], [50]). As for Assumption 5, it intuitively illustrates that the gap in step sizes among players is not large (cf., [19]).

Lemma 3 ([51]): Let $U(k)$, $\xi(k)$, and $\varphi(k)$ be deterministic nonnegative sequences such that

$$U(k+1) \leq U(k) + \xi(k) - \varphi(k),$$

with $\sum_{k=0}^{\infty} \xi(k) < \infty$. Then the sequence $U(t)$ converges and $\sum_{k=0}^{\infty} \varphi(k) < \infty$.

Since $Q_i \succ 0$, $J_i^F(x_i, \sigma_i(\boldsymbol{x}_{-i}), y)$ is strong convex over Ω_i with respect to x_i and strong convexity parameter is denoted as C_i'. Based on this, the convergence of Algorithm 1 is given by the following result.

Theorem 4: Suppose that Assumptions 2-5 hold.

(1) If the strong convex parameters C_i', C_2 and C satisfy $C_i' > \kappa \bar{L}$ and $\min(C_2, C) > \kappa \bar{T} \bar{L}$, where $\bar{L} = \max(2L, 2L_0, 2L_1)$ with L, L_0, L_1 defined in (19)-(21), then Algorithm 2 converges to a fixed point $(\boldsymbol{x}^\diamond, y^\diamond, \eta^\diamond)$;

(2) Moreover, if $(y^\diamond, \eta^\diamond)$ in the convergent point is the stationary point of $\Pi_2(y, \eta, \sigma_0(\boldsymbol{x}^\diamond))$ in (15), then $(\boldsymbol{x}^\diamond, y^\diamond)$ is the NE point of the non-convex network aggregative game (5).

Proof: We first prove that Algorithm 2 converges to the point $(\boldsymbol{x}^\diamond, y^\diamond, \eta^\diamond)$ satisfying $x_i^\diamond = P_{\Omega_i}\left(x_i^\diamond - \alpha_i^t g_i(x_i^\diamond, \sigma_i^\diamond, y^\diamond)\right)$, where $\sigma_i^\diamond = \sigma_i(\boldsymbol{x}_{-i}^\diamond)$. Let us introduce the notation $\Delta x_i^t = x_i^t - x_i^\diamond$, $\Delta y^t = y^t - y^\diamond$ and $\Delta \eta^t = \eta^t - \eta^\diamond$. Since the projection operator $P_{\Omega_i}(\cdot)$ is nonexpansive, we have

$$\begin{aligned} \left\| \Delta x_i^{t+1} \right\|^2 &\leq \left\| \Delta x_i^t - \alpha_i^t \left(g_i(x_i^t, \sigma_i^t, y^t) - g_i(x_i^\diamond, \sigma_i^\diamond, y^\diamond)\right) \right\|^2 \\ &\leq \left\| \Delta x_i^t \right\|^2 + 4A_i^2 \left(\alpha_i^t\right)^2 - 2\alpha_i^t \Psi_i^t, \end{aligned} \tag{25}$$

where $\Psi_i^t = \left(g_i(x_i^t, \sigma_i^t, y^t) - g_i(x_i^\diamond, \sigma_i^\diamond, y^\diamond)\right)^T \Delta x_i^t$, $\| g_i(x_i^t, \sigma_i^t, y^t) \| \leq A_i$, and $\sigma_i^t = \sigma_i(\boldsymbol{x}_{-i}^t)$.

Denote $\Delta c^t = \mathrm{col}\{\Delta y^t, \Delta \eta^t\}$. Considering the leader's and canonical dual variable's strategy at iteration t_j^L, and following the same operation as (25), we have

$$\begin{aligned} \left\| \Delta c^{t_j^L+1} \right\|^2 &= \left\| \begin{bmatrix} \Delta y^{t_j^L+1} \\ \Delta \eta^{t_j+1} \end{bmatrix} \right\|^2 \\ &\leq \left\| \Delta y^{t_j^L} \right\|^2 + 4A_y^2 \left(\beta^{t_j^L}\right)^2 - 2\beta^{t_j^L} \Psi_y^{t_j^L} \\ &\quad + \left\| \Delta \eta^{t_j^L} \right\|^2 + 4A_\eta^2 \left(\gamma^{t_j^L}\right)^2 - 2\gamma^{t_j^L} \Psi_\eta^{t_j^L}, \end{aligned} \tag{26}$$

where $\Psi_y^t = \left(g_y(y^t, \sigma_0^t, \eta^t) - g_y(y^\diamond, \sigma_0^\diamond, \eta^\diamond)\right)^T \Delta y^t$ and $\Psi_\eta^t = \left(g_\eta(y^t, \sigma_0^t, \eta^t) - g_\eta(y^\diamond, \sigma_0^\diamond, \eta^\diamond)\right)^T \Delta \eta^t$, $\sigma_0^t = \sigma_0(\boldsymbol{x}^t)$, $\| g_y(y^t, \sigma_0^t, \eta^t) \| \leq A_y$, $\| g_\eta(y^t, \sigma_0^t, \eta^t) \| \leq A_\eta$.

Now, let us introduce

$$\Sigma^j = \left\| \Delta c^{t_{j-1}^L+1} \right\|^2 + \sum_{i \in \mathcal{V}} \left\| \Delta x_i^{t_{j-1}^L+1} \right\|^2.$$

Therefore, using inequalities (25) and (26) for $t = t_{j-1}^L, \cdots, t_j^L$, we have

$$\begin{aligned}\Sigma^{j+1} \leq & \Sigma^j + 4A_y^2\left(\beta^{t_j^L}\right)^2 - 2\beta^{t_j^L}\Psi_y^{t_j^L} \\ & + 4A_\eta^2\left(\gamma^{t_j^L}\right)^2 - 2\gamma^{t_j^L}\Psi_\eta^{t_j^L} \\ & + 4\sum_{i\in\mathcal{V}}A_i^2\sum_{t'\in\mathcal{T}_j'}\left(\alpha_i^{t'}\right)^2 - 2\sum_{i\in\mathcal{V}}\sum_{t'\in\mathcal{T}_j'}\alpha_i^{t'}\Psi_i^{t'}, \quad (27)\end{aligned}$$

where $\mathcal{T}_j' = \left\{t_{j-1}^L + 1, \ldots, t_j^L\right\}$. Based on the definition of $\Psi_i^{t'}$ and adding and subtracting the term $g_i\left(x_i^\diamond, \sigma_i^{t'}, y^{t'}\right)$, we obtain

$$\begin{aligned}\Psi_i^{t'} = & \left(g_i\left(x_i^{t'}, \sigma_i^{t'}, y^{t'}\right) - g_i\left(x_i^\diamond, \sigma_i^{t'}, y^{t'}\right)\right)^T \Delta x_i^{t'} \\ & + \left(g_i\left(x_i^\diamond, \sigma_i^{t'}, y^{t'}\right) - g_i\left(x_i^\diamond, \sigma_i^\diamond, y^\diamond\right)\right)^T \Delta x_i^{t'}. \quad (28)\end{aligned}$$

By (19), we have

$$\begin{aligned}-\alpha_i^{t'}\Psi_i^{t'} \leq & -\alpha_i^{t'}C_i'\left\|\Delta x_i^{t'}\right\|^2 \\ & + \alpha_i^{t'}L\left(\left\|\Delta y^{t'}\right\| + \left\|\sigma_i^{t'} - \sigma_i^\diamond\right\|\right)\left\|\Delta x_i^{t'}\right\| \\ \leq & -\mu_{\min}^{t'}C_i'\left\|\Delta x_i^{t'}\right\|^2 \\ & + \kappa\mu_{\min}^{t'}L\left\|\Delta x_i^{t'}\right\| \cdot \left(\left\|\Delta y^{t'}\right\| + \sum_{j\in\mathcal{N}_i}a_{ij}\left\|\Delta x_j^{t'}\right\|\right). \quad (29)\end{aligned}$$

Recall that concerning the variable y, $\Pi_2(\cdot, \eta, \sigma_0(\boldsymbol{x}))$ exhibits strongly convexity over Ω_y when $\eta \in \mathcal{S}_2^+$ and the relation in (20). We shall adopt the same procedure as (29) and arrive at:

$$\begin{aligned}& -\beta^{t_j^L}\Psi_y^{t_j^L} \\ & \leq -\beta^{t_j^L}C_2\left\|\Delta y^{t_j^L}\right\|^2 \\ & \quad + \beta^{t_j^L}\left(L_1\left\|\Delta\eta^{t_j^L}\right\| + L_0\left\|\sigma_0^{t_j^L} - \sigma_0^\diamond\right\|\right)\left\|\Delta y^{t_j^L}\right\| \\ & \leq -\mu_{\min}^{t_j^L}C_2\left\|\Delta y^{t_j^L}\right\|^2 \\ & \quad + \kappa\mu_{\min}^{t_j^L}L_0\left\|\Delta y^{t_j^L}\right\| \cdot \left(\left\|\Delta\eta^{t_j^L}\right\| + \sum_{i\in\mathcal{V}}a_{0i}\left\|\Delta x_i^{t_j^L}\right\|\right). \quad (30)\end{aligned}$$

By following the same procedure as (29) and (30), for the canonical dual variable η, we have

$$\begin{aligned}& -\gamma^{t_j^L}\Psi_\eta^{t_j^L} \\ & \leq -\gamma^{t_j^L}C\left\|\Delta\eta^{t_j^L}\right\|^2 \\ & \quad + \gamma^{t_j^L}L_1\left(\left\|\Delta y^{t_j^L}\right\| + \left\|\sigma_0^{t_j^L} - \sigma_0^\diamond\right\|\right)\left\|\Delta\eta^{t_j^L}\right\| \\ & \leq -\mu_{\min}^{t_j^L}C\left\|\Delta\eta^{t_j^L}\right\|^2 \\ & \quad + \kappa\mu_{\min}^{t_j^L}L_1\left\|\Delta\eta^{t_j^L}\right\| \cdot \left(\left\|\Delta y^{t_j^L}\right\| + \sum_{i\in\mathcal{V}}a_{0i}\left\|\Delta x_i^{t_j^L}\right\|\right). \quad (31)\end{aligned}$$

Therefore, substituting inequalities (29), (30) and (31) into (27), it can be concluded that

$$\begin{aligned}& -\beta^{t_j^L}\Psi_y^{t_j^L} - \gamma^{t_j^L}\Psi_\eta^{t_j^L} - \sum_{i\in\mathcal{V}}\sum_{t'\in\mathcal{T}_j'}\alpha_i^{t'}\Psi_i^{t'} \\ & \leq -\mu_{\min}^{t_j^L}C_2\left\|\Delta y^{t_j^L}\right\|^2 - \mu_{\min}^{t_j^L}C\left\|\Delta\eta^{t_j^L}\right\|^2 \\ & \quad - \sum_{i\in\mathcal{V}}\sum_{t'\in\mathcal{T}_j'}\mu_{\min}^{t'}C_i'\left\|\Delta x_i^{t'}\right\|^2 + \kappa\sum_{t'\in\mathcal{T}_j'}\mu_{\min}^{t'}z^{t'^T}\mathcal{Z}^{t'}z^{t'}, \quad (32)\end{aligned}$$

where $z^{t'} = \mathrm{col}\left\{\left\|\Delta x_1^{t'}\right\|, \ldots, \left\|\Delta x_N^{t'}\right\|, \left\|\Delta y^{t'}\right\|, \left\|\Delta\eta^{t'}\right\|\right\}$,

$$\mathcal{Z}^{t'} = \begin{cases}\begin{bmatrix} L\mathcal{A} & L\mathbf{1}_N & 0 \\ L_0a_0^T & 0 & L_0 \\ L_1a_0^T & L_1 & 0\end{bmatrix} & , t' \in \mathcal{T}^L \\ \begin{bmatrix} L\mathcal{A} & L\mathbf{1}_N & 0 \\ 0_N^T & 0 & 0 \\ 0_N^T & 0 & 0\end{bmatrix} & , t' \notin \mathcal{T}^L.\end{cases} \quad (33)$$

Here, $\mathbf{1}_N$ represents the N-dimensional column vector of unit entries. An apparent induction gives that the row sum of each row in the matrices can be $2L$, $2L_0$, or $2L_1$. Consequently, according to Perron-Frobenius Theorem [52], $\| \mathcal{Z}^{t'} \| \leq \max(2L, 2L_0, 2L_1) = \bar{L}$ for $t' \in \mathcal{T}^L$ and $\| \mathcal{Z}^{t'} \| \leq 2L$ for $t' \notin \mathcal{T}^L$, respectively.

Hence, $z^{t'^T}\mathcal{Z}^{t'}z^{t'} \leq \bar{L} \| z^{t'} \|^2$ for any $t' \geq 0$. Therefore,

$$\begin{aligned}\sum_{t'\in\mathcal{T}_j'}\mu_{\min}^{t'}z^{t'^T}\mathcal{Z}^{t'}z^{t'} \leq & \sum_{t'\in\mathcal{T}_j'}\mu_{\min}^{t'}\bar{L}\left(\sum_{i\in\mathcal{V}}\left\|\Delta x_i^{t'}\right\|^2\right) \\ & + \mu_{\min}^{t_j^L}\left(t_j^L - t_{j-1}^L\right)\bar{L}\left(\left\|\Delta y^{t_j^L}\right\|^2 + \left\|\Delta\eta^{t_j^L}\right\|^2\right), \quad (34)\end{aligned}$$

where the last term of (34) is rearranged. Since the leader and the canonical dual variable only update their variables at $t_j^L \in \mathcal{T}^L$. Thus, the terms $\left\|\Delta y^{t_j^L}\right\|$ and $\left\|\Delta\eta^{t_j^L}\right\|$ are the same from iteration $t_{j-1}^L \in \mathcal{T}^L$ to $t_j^L \in \mathcal{T}^L$. By substituting (32) and (34) into (27), it can be concluded that

$$\begin{aligned}& \Sigma^{j+1} \\ & \leq \Sigma^j + 4A_y^2\left(\beta^{t_j^L}\right)^2 + 4A_\eta^2\left(\gamma^{t_j^L}\right)^2 + \sum_{i\in\mathcal{V}}4A_i^2\sum_{t'\in\mathcal{T}_j'}\left(\alpha_i^{t'}\right)^2 \\ & \quad - 2\mu_{\min}^{t_j^L}\left(C_2 - \kappa\left(t_j^L - t_{j-1}^L\right)\bar{L}\right)\left\|\Delta y^{t_j^L}\right\|^2 \\ & \quad - 2\mu_{\min}^{t_j^L}\left(C - \kappa\left(t_j^L - t_{j-1}^L\right)\bar{L}\right)\left\|\Delta\eta^{t_j^L}\right\|^2 \\ & \quad - 2\sum_{i\in\mathcal{V}}\sum_{t'\in\mathcal{T}_j'}\mu_{\min}^{t'}(C_i' - \kappa\bar{L})\left\|\Delta x_i^{t'}\right\|^2 \\ & \triangleq \Sigma^j + I_1^j + I_2^j + I_3^j - I_4^j - I_5^j - I_6^j.\end{aligned}$$

Based on Assumption 4, $\sum_{j=0}^{\infty} I_1^j + I_2^j + I_3^j$ is bounded. According to Assumption 3, $C_2 - \kappa\left(t_j^L - t_{j-1}^L\right)\bar{L} \geq C_2 - \kappa\bar{T}\bar{L} > 0$, $C - \kappa\left(t_j^L - t_{j-1}^L\right)\bar{L} \geq C - \kappa\bar{T}\bar{L} > 0$, $C_i' - \kappa\bar{L} > 0$. Therefore, I_4^j, I_5^j, and I_6^j are positive. Now, the conditions of Lemma 3 are satisfied, and consequently, $\sum_{j=0}^{\infty} I_4^j + I_5^j + I_6^j <$

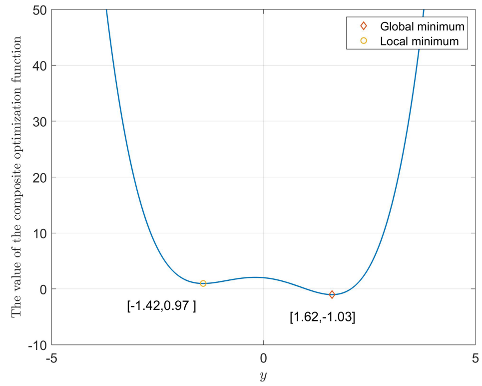

Fig. 1. The composite optimization function.

∞ converges. Due to the positiveness of I_4^j, I_5^j, and I_6^j, $\sum_{j=0}^{\infty} I_4^j < \infty$, $\sum_{j=0}^{\infty} I_5^j < \infty$, and $\sum_{j=0}^{\infty} I_6^j < \infty$ are concluded. Therefore, based on $\sum_{t=0}^{\infty} \mu_{\min}^t \geq \frac{1}{\kappa} \sum_{t=0}^{\infty} \mu_{\max}^t = \infty$, both $\| \Delta x_i^{t'} \|$, $\| \Delta \eta_j^{t_i^L} \|$ and $\| \Delta y^{t_i^L} \|$ converge to 0. Thus, x_i^t and y^t converge to $x_i^\diamond$ and $y^\diamond$. Furthermore, by Theorem 3, Algorithm 2 converges to the NE. □

VI. Simulation

In this section, we provide an example of sensor network localization tasks (cf., [32]) to illustrate the performance of our proposed algorithms in this paper (i.e., Algorithm 1 and Algorithm 2).

In a sensor network, multiple sensors (followers) $\mathcal{V} = \{1, \cdots, 5\}$ are connected to the aggregator (leader). Let us define $x_i \in \mathbb{R}$ and $y \in [-2, 2]$ as the locations of the i-th follower and the leader, i.e., the strategies of the i-th follower and the leader, respectively. The cost of every follower $i \in \mathcal{V}$ is set to (2), where $Q_i = i > 0$, $P_{0i} = 1$, which represents the energy of the follower. The followers are connected through a weighted directed graph with $\mathcal{A} = \begin{bmatrix} 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{8} & \frac{7}{8} \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

On the other hand, the distance between the leader and the average location of all followers is given by $d = 2$. The leader aims to finding its location such that $\|y - \sigma_0(\boldsymbol{x})\|^2 = d$. When the measurement contains noise, the leader's cost function can be formulated as

$$J^L(y, \sigma_0(\boldsymbol{x})) = \frac{1}{2}(\frac{1}{2}\|y - \sigma_0(\boldsymbol{x})\|^2 - d)^2 - \frac{1}{2}(y - \sigma_0(\boldsymbol{x}))^T.$$

Here, the last term is a deviation term to guarantee the uniqueness of equilibrium, similar to the treatment in [31] and [32]. This term does not affect the nonconvexity.

(1) We first compute the SE point by Algorithm 1 since it is clear that the augmented matrix Q is invertible and then followers' best response strategies set to the aggregator's strategy is obtained as $BR(y) = -Q^{-1}Py$ with $P = [1, 1, 1, 1, 1]^T$. Following the procedure presented in Section IV, Fig.1 plots the composite optimization function (7), which is non-convex with the local minimum $y = -1.42$ and the global minimum $y^\diamond = 1.62$. Then we get the total complementarity function

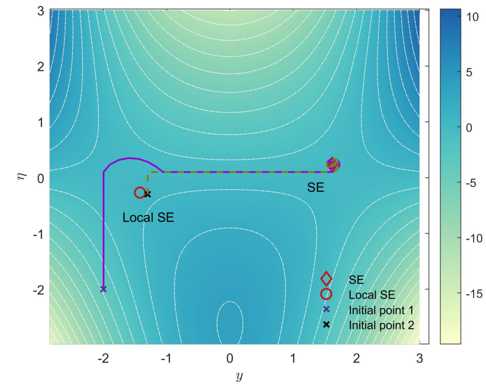

Fig. 2. Convergence results with different initial points by Algorithm 1.

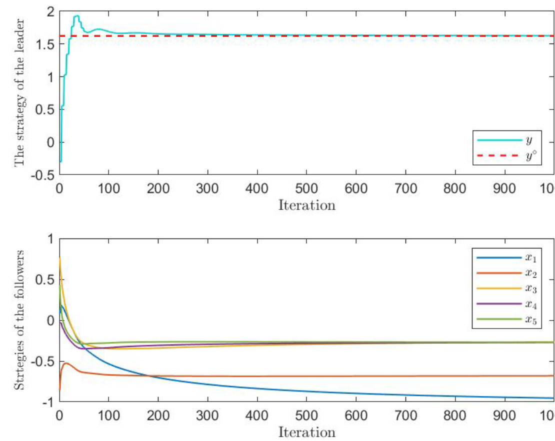

Fig. 3. Convergence of players' strategies to the NE by Algorithm 2.

for this non-convex optimization problem as follows

$$\Pi_1(y, \eta) = \frac{1}{2}\|zy\|^2\eta - \frac{1}{2}\eta^2 - d\eta - \frac{1}{2}(zy)^T,$$

where $z = 1 + (a_0^T \otimes I_n)Q^{-1}P$.

We take $\mathcal{S}_1^+ = \{\eta \in [-2, 1.42] \mid \eta \geq 0.1\} = [0.1, 1.42]$ and set the constant step size as $\tau = 0.1$. Fig. 2 draws the trajectories of the decision variable of the leader by starting from the different initial points, from which we can see that even though the initial values are in the local neighborhood of the local minimum, the sequence $\{y^t\}$ is within their local constraint sets and converges to the global minimum $y^\diamond = 1.617$. Then, by Step 3, we compute $\boldsymbol{x}^\diamond = \text{col}\{-1.04, -0.67, -0.23, -0.26, -0.28\}$. At last, the players arrive at the SE point $(\boldsymbol{x}^\diamond, y^\diamond)$.

(2) However, the loss of the aggregator's leader position may occur in certain scenarios such as when the aggregator (leader) may incur observation errors. Here, NE will be what all players are committed to achieving. We then compute the NE point by Algorithm 2. For each follower i, let the location constraint set $\Omega_i = [-1, 1]$. Here, we assume that the leader makes the decision once every ten iterations. In Fig. 3, we plot the trajectories of the leader's strategy y^t and followers' strategies $\boldsymbol{x}^t$ with initial values $\boldsymbol{x}^0 = [0, -1, 1, 0, 1]^T$ and $y^0 = -0.3$. It is clear that the strategies of the leader and followers satisfy their local set constraints and converge to the NE $(\boldsymbol{x}^\diamond, y^\diamond) = \text{col}\{-1, -0.67, -0.24, -0.27, -0.28, 1.62\}$.

In addition, we compare Algorithm 2 with the projected gradient descent (PGD) method [21]. Fig. 4 shows the performance of two algorithms starting from the initial point

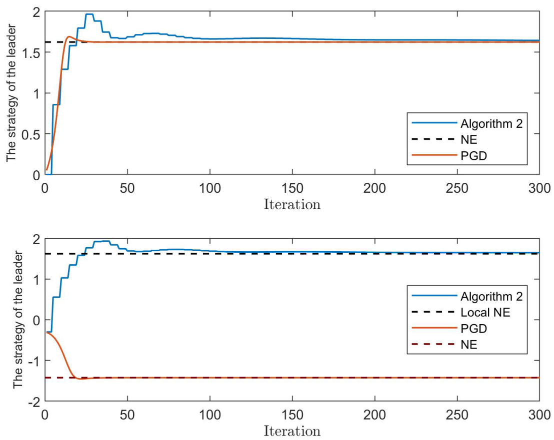

Fig. 4. Performance of different algorithms for NE computation.

$y^0 = 0$ far from the local NE (the upper one) and the initial point $y^0 = -0.3$ close to the local NE (the lower one). It indicates that the PGD may get trapped in the local NE, while our algorithm always converges to the NE in the non-convex case irrespective of its initial value, which again illustrates the effectiveness of our algorithm.

VII. Concluding Remarks

This paper investigated the SE and NE computation in the non-convex network aggregative game with one leader and multiple followers, where the cost functions of the leader and followers are non-convex and strongly convex-quadratic, respectively. By virtue of canonical duality theory, discrete projected gradient algorithms are proposed, and the convergence to the global equilibria of the non-convex game is proved. Many intriguing questions are worthy of further investigation. For example, as for algorithm design, distributed protocols can be explored if privacy and security are taken into account. In terms of the game setup, our formulation is based on a static game. Dynamic games can also be studied in a non-convex setting, considering the complex application context. Moreover, searching for equilibria in general non-convex games is still an open problem due to the diversity of non-convex structures.

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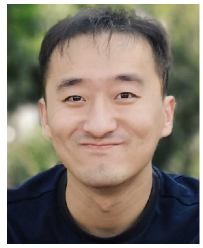
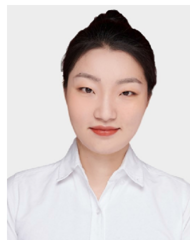
Rongjiang Li received the B.S. degree in statistics from Wuhan University, Wuhan, China, in 2019. She is currently pursuing the Ph.D. degree in systems theory with the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China. Her current research interests include multi-agent systems, game theory, and system identification.



Guanpu Chen received the B.Sc. degree in mathematics from the University of Science and Technology of China, Hefei, China, and the Ph.D. degree in systems science from the Chinese Academy of Sciences, Beijing, China. He is currently a Post-Doctoral Researcher with the School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Stockholm, Sweden. His research interests include operational research and cybernetics, including multi-agent systems, distributed optimization, network games, and machine learning.



Die Gan (Member, IEEE) received the B.S. degree in mathematics from Shandong University, Jinan, China, in 2017, and the Ph.D. degree in systems theory from the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2022. She is currently a Post-Doctoral Fellow with the Zhongguancun Laboratory, Beijing, China. Her research interests include system identification, game theory, distributed control, and distributed adaptive filter. She received the China National Scholarship for graduate students in 2020 and the Outstanding Graduate Title of Beijing in 2022.

Haibo Gu (Member, IEEE) received the M.Sc. degree in mathematics from Inner Mongolia University, Hohhot, China, in 2016, and the Ph.D. degree in system theory from the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2019. From 2019 to 2021, he was a Post-Doctoral Fellow of the Zhuoyue Program, Beihang University, Beijing, where he is currently an Associate Professor with the School of Automation Science and Electrical Engineering. His current research interests include complex networks, multi-agent systems, network dynamics, and cooperative control.

Jinhu Lü (Fellow, IEEE) received the Ph.D. degree in applied mathematics from the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2002.

He was a Professor with RMIT University, Melbourne, VIC, Australia, and a Visiting Fellow with Princeton University, Princeton, NJ, USA. Currently, he is the Dean of the School of Automation Science and Electrical Engineering, Beihang University, Beijing. He is also a Professor with the Academy of Mathematics and Systems Science, Chinese Academy of Sciences. He is also a Chief Scientist of the National Key Research and Development Program of China and a Leading Scientist of Innovative Research Groups of the National Natural Science Foundation of China. His current research interests include complex networks, industrial internet, network dynamics, and cooperation control. He served as a member of the Fellows Evaluating Committee for the IEEE CASS, the IEEE CIS, and the IEEE IES. He is a fellow of CAA. He was a recipient of the prestigious Ho Leung Ho Lee Foundation Award in 2015; the National Innovation Competition Award in 2020; the State Natural Science Award three times from the Chinese Government in 2008, 2012, and 2016; the Australian Research Council Future Fellowships Award in 2009; the National Natural Science Fund for Distinguished Young Scholars Award; and the Highly Cited Researcher Award in engineering from 2014 to 2020. He was the General Co-Chair of IECON in 2017. He is/was an Editor of various ranks for 15 SCI journals, including the Co-Editor-in-Chief of IEEE TRANSACTIONS ON INDUSTRIAL INFORMATICS.